

Maximum occupation time of a transient excited random walk on \mathbb{Z}

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Abstract

We consider a transient excited random walk on \mathbb{Z} and study the asymptotic behavior of the occupation time of a currently most visited site. In particular, our results imply that, in contrast to the random walks in random environment, a transient excited random walk does not spend an asymptotically positive fraction of time at its favorite (most visited up to a date) sites.

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1 Introduction and statement of results

Excited random walks or random walks in a cookie environment on \mathbb{Z}^d is a modification of the nearest neighbor simple random walk such that in several first visits to each site of the integer lattice, the walk's jump kernel gives a preference to a certain direction and assigns equal probabilities to the remaining $(2d - 1)$ directions. If the current location of the random walk has been already visited more than a certain number of times, then the walk moves to one of its nearest neighbors with equal probabilities. The model was introduced by Benjamini and Wilson in [3] and extended by Zerner in [12].

In this paper we focus on the excited random walks in dimension one. To define the transition mechanism of the random walk, fix an integer $M \in \mathbb{N}$ and let

$$\Omega_M = \left\{ \omega(z, i)_{i \in \mathbb{N}, z \in \mathbb{Z}} : \omega(z, i) \in [0, 1], \forall 1 \leq i \leq M, \text{ and } \omega(z, i) = 0.5 \text{ otherwise} \right\}.$$

The value of $\omega(z, i)$ determines the probability of the jump from z to $z + 1$ upon i -th visit of the random walk to the site $z \in \mathbb{Z}$. The random walk is assumed to be nearest neighbor, and hence the probability of the jump from z to $z - 1$ upon its i -th visit to z is given by the complementary probability $1 - \omega(z, i)$. The elements of the set Ω_M are called *cookie environments*.

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For a fixed $z \in \mathbb{Z}$, $(\omega(z, i))_{1 \leq i \leq M}$ can be thought of as a sequence of numerical characteristics called “strengthes”, associated with a pile of M “cookies” placed at z . Correspondingly, $\omega(z, i)$ is referred to as the *strength of the i -th cookie* at the pile. With a slight abuse of language, using the above introduced jargon we will often identify the strength $\omega(z, i)$ of a cookie with the cookie itself. Transition kernel of the random walk can be informally described as follows: while the supply of the cookies at a given site lasts, the walker eats a cookie upon each visit there and then makes one step in a random direction, such that the probability of moving to the right is equal to the “strength” of the just eaten cookie.

More precisely, the random walk in a cookie environment $\omega \in \Omega_M$ is defined as follows. Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, let $\Sigma = \mathbb{Z}^{\mathbb{N}_0}$ be the state space of the infinite paths of a discrete-time random walk on \mathbb{Z} (\mathbb{Z} for the location and \mathbb{N}_0 for the time), and let \mathcal{F} be its Borel σ -algebra (i. e., the σ -algebra generated by the cylinder sets of the infinite product space Σ). For any $x \in \mathbb{Z}$ and $\omega \in \Omega_M$, an excited random walk (abbreviated in what follows as ERW) starting at $x \in \mathbb{Z}$ in the cookie environment ω , is a sequence of random variables $X = (X_n)_{n \in \mathbb{N}_0}$ defined in a probability space $(\Sigma, \mathcal{F}, P_{x, \omega})$ such that $P_{x, \omega}(X_0 = x) = 1$ and

$$P_{x, \omega}(X_{n+1} = X_n + 1 | \mathcal{F}_n) = \omega(X_n, \xi_n),$$

where $\mathcal{F}_n := \sigma(X_i, 0 \leq i \leq n)$ and

$$\xi_n := \#\{0 \leq i \leq n : X_i = X_n\}. \quad (1)$$

The measure $P_{x, \omega}$ is usually referred to as the *quenched law* of the excited random walk in the cookie environment ω . Let \mathbb{P} be a probability measure on Ω_M that makes the collection of “piles” $\omega_z := \omega(z, i)_{i \in \mathbb{N}}$ indexed by $z \in \mathbb{Z}$ into an i.i.d. sequence. Notice that we do not insist on the independence of the cookies within a given pile, that is the random variables $\omega(z, i)$ for a fixed $z \in \mathbb{Z}$ can be dependent under \mathbb{P} . The (associated with \mathbb{P}) *annealed* (average) law P_x of the ERW on (Σ, \mathcal{F}) is defined by setting $P_x(\cdot) = \mathbb{E}[P_{x, \omega}(\cdot)]$, where \mathbb{E} is the expectation induced by the probability law \mathbb{P} .

Many important aspects of the asymptotic behavior of excited random walks on \mathbb{Z} are by now well-understood. In particular, Zerner in [12], Basdevant and Singh in [1, 2], Kosygina and Zerner in [10], and Kosygina and Mountford in [9] characterized the recurrence-transience behavior and possible speed regimes of the ERW, and proved limit theorems for the fluctuations of the current location of ERW. The goal of this paper is to study the asymptotic dynamics of the occupation time of a currently most visited site for a transient ERW.

Notice that the consumption of a cookie $\omega(z, i)$ results in creating the local drift (i.e., the bias in the conditional on the history \mathcal{F}_n expectation of the subsequent displacement) equal to $E_{\omega, x}[X_{n+1} - X_n | X_n = z, \xi_n = i] = 2\omega(z, i) - 1$. Let

$$\delta = \mathbb{E}\left[\sum_{i=1}^M (2\omega(z, i) - 1)\right]$$

be the annealed expectation of the total drift (i. e., the “boost” in the positive direction) available to the random walk at a site $z \in \mathbb{Z}$. Notice that since ω_z is assumed i. i. d., δ is independent of z . Following [10] and [9] (and in contrast to the original version presented in [3, 12]) we do not impose the condition that $\mathbb{P}(\omega(z, i) \geq 1/2) = 1$, that is allowing both

“positive” and “negative” cookies. It turns out (see [9, 10]) that the asymptotic behavior of an one-dimensional excited random walk is largely determined by the value of the parameter δ . In particular, under a mild non-degeneracy assumption on the cookie environment, the random walk is transient to the right if and only if $\delta > 1$.

Throughout this paper we will impose the following conditions on the cookie environment.

Assumption 1.1. *The following assumptions hold:*

- (a) *Independence:* the sequence of “piles” $\omega_z = (\omega(z, i))_{i \in \mathbb{N}}$ indexed by sites $z \in \mathbb{Z}$ is an i.i.d. sequence under \mathbb{P} .
- (b) *Non-degeneracy:* $\mathbb{E}[\prod_{i=1}^M \omega(0, i)] > 0$ and $\mathbb{E}[\prod_{i=1}^M (1 - \omega(0, i))] > 0$.
- (c) *Transience:* $\delta > 1$.

The above non-degeneracy condition ensures that, a priori, the random walk can at any given step move to each direction with a positive probability. It is known [10, 12] that under Assumption 1.1 the ERW is transient to the right (that is, $P_0(\lim_{n \rightarrow \infty} X_n = \infty) = 1$) and, furthermore, has the asymptotic speed

$$v := \lim_{n \rightarrow \infty} \frac{X_n}{n} \in [0, 1), \quad (2)$$

which is strictly positive if and only if $\delta > 2$.

Define the occupation time of the ERW at site $x \in \mathbb{Z}$ as

$$\xi_n(x) := \#\{0 \leq i \leq n : X_i = x\}. \quad (3)$$

Thus $\xi_n(x)$ is the number of times that the ERW visits $x \in \mathbb{Z}$ during the first n steps. Let

$$\xi_n^* := \max_{x \in \mathbb{Z}} \xi_n(x) \quad (4)$$

be the largest number of visits to a single site during the first n steps. For the sake of notational convenience we will occasionally write $\xi^*(n)$ for ξ_n^* . The asymptotic properties of the process $\xi^* := (\xi_n^*)_{n \in \mathbb{N}}$ can be compared to those of the simple random walk as well as of a random walk in random environment (abbreviated in what follows as RWRE) in dimension one. For a comprehensive up to date review of the latter topics, see a monograph of Révész [11]. For more recent developments on RWRE, we refer to [6, 8] and references therein.

This paper is devoted to the study of the limit points of the sequence ξ^* for a transient ERW. The approach adopted here is based on a reduction of the study of the asymptotic behavior of the ERW to that of a branching process and the subsequent reformulation of the problem in terms of the asymptotic dynamics of the most populated generation of the branching process. Similar questions for transient RWRE have been addressed in the interesting paper by Gantert and Shi [8]. The essential branching processes machinery which enables the implementation of our approach to excited random walks was introduced in [1, 2] and further developed in [9, 10].

Our first result concerns non-ballistic ERW.

Theorem 1.2. *Suppose that Assumption 1.1 holds with $\delta \in (1, 2]$. Then,*

(i) The following holds:

$$\limsup_{n \rightarrow \infty} \frac{\xi_n^*}{n^{1/2}} > 0, \quad P_0 - \text{a. s.} \quad (5)$$

(ii) Furthermore, for any $\alpha > \frac{1}{\delta}$:

$$\lim_{n \rightarrow \infty} \frac{\xi_n^*}{n^{1/2}(\log n)^\alpha} = 0 \quad \text{while} \quad \lim_{n \rightarrow \infty} \frac{(\log n)^\alpha \xi_n^*}{n^{1/2}} = \infty, \quad P_0 - \text{a. s.}$$

The above theorem implies in particular that unlike RWRE (see [8]), a non-ballistic transient ERW does not spend a positive fraction of time at its favorite sites. While the asymptotic behavior of ξ_n^* for transient RWRE seems to be determined by the so called “traps” created by a random potential (cf. [8] and [6]), and is radically different from that of the simple unbiased random walk, the limsup asymptotic of ξ_n^* for a non-ballistic transient ERW turns out to be rather similar to its counterpart for a simple non-biased random walk (cf. Theorem 11.3 in [11]). We also remark that, based on a comparison with the latter, we believe that in fact $\limsup_{n \rightarrow \infty} \frac{\xi_n^*(n)}{n^{1/2}} = \infty$ under the conditions of Theorem 1.2, but were unable to prove this conjecture.

The next theorem deals with the asymptotic behavior of $\xi^*(n)$ for ballistic ERW. Recall that, for $\rho \in \mathbb{R}$, a sequence $\phi : \mathbb{N} \rightarrow \mathbb{R}$ is *regularly varying (at infinity) of index ρ* if $f(n) = n^\rho L(n)$ for some sequence $L : \mathbb{N} \rightarrow \mathbb{R}$ such that $\lim_{n \rightarrow \infty} L(\lfloor \lambda n \rfloor)/L(n) = 1$ for all reals $\lambda > 0$ (that is $L(n)$ is *slowly varying*). Here and henceforth, $\lfloor x \rfloor$ denotes the integer part of a real number x , that is $\lfloor x \rfloor = \max\{n \in \mathbb{N} : n \leq x\}$.

Theorem 1.3. *Suppose that Assumption 1.1 holds with $\delta > 2$.*

(i) *Let $\phi : \mathbb{N} \rightarrow \mathbb{R}$ be a regularly varying sequence of a positive index $\rho > 0$. Then,*

$$\limsup_{n \rightarrow \infty} \frac{\xi_n^*}{\phi(n)} = \begin{cases} 0 \\ \infty \end{cases} \quad P_0 - \text{a. s.} \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \frac{1}{[\phi(n)]^\delta} \begin{cases} < \infty \\ = \infty. \end{cases}$$

In particular, for any $\alpha > 1/\delta$,

$$\lim_{n \rightarrow \infty} \frac{\xi_n^*}{n^{1/\delta}(\log n)^\alpha} = 0, \quad P_0 - \text{a. s.},$$

and hence $\lim_{n \rightarrow \infty} \frac{\xi_n^}{n} = 0$, $P_0 - \text{a. s.}$*

(ii) *For $\alpha > 1/\delta$,*

$$\lim_{n \rightarrow \infty} \frac{(\log n)^\alpha \xi_n^*}{n^{1/\delta}} = \infty, \quad P_0 - \text{a. s.}$$

For the sake of comparison with the second part of Theorem 1.2 we next explicitly state the following:

Corollary 1.4. *Under the conditions of Theorem 1.3, the following holds for any $\alpha > \frac{1}{\delta}$:*

$$\lim_{n \rightarrow \infty} \frac{\xi_n^*}{n^{1/\delta} (\log n)^\alpha} = 0 \quad \text{while} \quad \lim_{n \rightarrow \infty} \frac{(\log n)^\alpha \xi_n^*}{n^{1/\delta}} = \infty, \quad P_0 - \text{a. s.}$$

The rest of the paper is organized as follows. In Section 2, we consider an auxiliary branching process formed by successive level crossings along the random walk path. The proof of Theorems 1.2 and 1.3 are contained in Section 3.

2 Reduction to a branching process

The proof of our main results which is given in the next section relies on the use of a mapping of the paths of ERW into realizations of a suitable branching process with migration. In this section we discuss the branching process framework and recall some auxiliary results related to it; see [9, 10] and [7] for more details.

For $m \in \mathbb{N}$, let T_m be the first hitting time of site m , that is

$$T_m = \inf \{n \in \mathbb{N} : X_n = m\}.$$

Since the ERW is transient to the right under Assumption 1.1, the random variables T_m are almost sure finite for all $m \in \mathbb{N}$ under the law P_0 . Moreover, it follows from (2) (by passing to the random subsequence of indexes $n = T_m$ in (2)) that

$$\lim_{n \rightarrow \infty} \frac{T_n}{n} = \lim_{m \rightarrow \infty} \frac{T_m}{X_{T_m}} = v^{-1} \in (0, \infty], \quad P_0 - \text{a. s.} \quad (6)$$

Set now $D_m^m := 0$ and for $k \leq m - 1$ let

$$D_k^m = \sum_{i=0}^{T_m-1} \mathbf{1}_{\{X_{i+1}=k-1, X_i=k\}}$$

be the number of down-crossing steps of the ERW from site k to $k - 1$ before time T_m . Then (see, for instance, [8, 10]),

$$T_m = m + 2 \sum_{k \leq m} D_k^m = m + 2 \sum_{0 \leq k \leq m} D_k^m + 2 \sum_{k < 0} D_k^m. \quad (7)$$

It follows from (3) that

$$\xi_{T_m}(k) = \begin{cases} 0 & \text{for } k > m \\ D_k^m + D_{k+1}^m + \mathbf{1}_{\{k \geq 0\}} & \text{for } k \leq m, \end{cases}$$

and hence

$$\begin{aligned} \max_{0 \leq k \leq m} D_k^m &\leq \xi_{T_m}^* = \max_{k < m} (D_k^m + D_{k+1}^m + \mathbf{1}_{\{k \geq 0\}}) \\ &\leq 1 + 2 \max_{0 \leq k \leq m} D_k^m + 2 \max_{k < 0} D_k^m. \end{aligned} \quad (8)$$

Notice that $\max_{k < 0} D_k^m$ is bounded above by the total time spent by the random walk on the negative half-line. Since the random walk is transient to the right, the latter quantity is P_0 – a.s. finite. Therefore, for any eventually increasing non-negative sequence $\phi : \mathbb{N} \rightarrow \mathbb{R}_+$, we have

$$\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq k \leq m} D_k^m}{\phi(m)} \leq \limsup_{m \rightarrow \infty} \frac{\xi_{T_m}^*}{\phi(m)} \leq 2 \limsup_{m \rightarrow \infty} \frac{\max_{0 \leq k \leq m} D_k^m}{\phi(m)}.$$

Similar inequalities hold with the \limsup replaced by the \liminf .

To elucidate the probabilistic structure of the sequence D_k^m , it is convenient to exploit the following alternative definition of the random walk $(X_n)_{n \geq 0}$. Assume that the underlying probability space is enlarged to include a sequence of, conditionally on ω , independent Bernoulli random variables (“coins”) $(B(z, i))_{i \in \mathbb{N}, z \in \mathbb{Z}}$ such that

$$P_{0, \omega}(B(z, i) = 1) = \omega(z, i) \quad \text{and} \quad P_{0, \omega}(B(z, i) = -1) = 1 - \omega(z, i). \quad (9)$$

Then the ERW X can be alternatively defined by specifying the jump sequence recursively, as follows:

$$X_{n+1} = X_n + B(X_n, \xi_n), \quad (10)$$

where ξ_n is introduced in (1). We adopt the terminology of [9, 10] and refer to the event $\{B(z, i) = 1\}$ as a “success” and to the event $\{B(z, i) = -1\}$ as a “failure”. For $z \geq 0$, denote by $F_m^{(z)}$ the number of failures before the m -th success in the sequence $B^{(z)} := (B(z, i))_{i \in \mathbb{N}}$. Let $V := (V_k)_{k \geq 0}$ be a Markov chain on \mathbb{N}_0 with transition kernel defined (under the law P_0) by means of the following recursion:

$$V_{k+1} = F_{V_k+1}^{(k)}, \quad k \geq 0.$$

The process V can be thought of as a branching process with the following properties:

1. There is exactly 1 immigrant in each generation and the immigration happens before the reproduction.
2. The number of offspring of m -th individual in generation $k \in \mathbb{N}$ is equal to $F_m^{(k)} - F_{m-1}^{(k)}$.

For non-negative reals $x \geq 0$, denote by P_x^V the law of the process V that starts with $[x]$ individuals in the generation zero. It turns out that for every $n \geq 0$, the distribution of (V_0, V_1, \dots, V_n) under P_0^V coincides with the distribution of the array $(D_n^n, D_{n-1}^n, \dots, D_0^n)$ associated with a transient to the right ERW (see, for instance, Section 2 in [9]).

Under Assumption 1.1, X is transient to the right, and hence there exists an infinite sequence of times when the random walk moves forward to a “fresh point”, i. e. to a site which has been never visited before and to which it will never return afterwards [10, 12]. It turns out [1, 10] that for the branching process V this implies that the following random times are finite with probability one under the law P_0^V :

$$\sigma_{-1} := 0 \quad \text{and} \quad \sigma_k := \inf\{i > \sigma_{k-1} : V_i = 0\}, \quad k \geq 0.$$

Thus $(\sigma_k)_{k \geq 0}$ is the sequence of renewal times in which the extinction occurs and the branching process starts afresh due to the immigration. Notice that while the immigrants serve as “founders of dynasties” of descendants, they themselves are not counted in the population of the branching process. In what follows we refer to the part of the branching process evolving between two successive extinction times as a *life cycle* of the process. The difference $\sigma_k - \sigma_{k-1}$, $k \geq 0$, represents therefore the duration of the $(k+1)$ -th life cycle. We remark that, although under our assumptions the event $\sigma_k - \sigma_{k-1} = 1$ can happen with a positive probability for any $k \geq 0$, the branching process does not get absorbed at zero, and is eventually revived in a future generation with a strictly positive number of immigrants. Let

$$\varrho_m := \min\{k \geq 0 : \sigma_k \geq m\} \quad (11)$$

denote the number of the renewal epochs completed by the first m generations.

Further, let

$$S_k := \sum_{i=\sigma_{k-1}}^{\sigma_k-1} V_i, \quad k \geq 0,$$

be the total population present during the $(k+1)$ -th cycle, and let

$$M_k := \max_{\sigma_{k-1} \leq i < \sigma_k} V_i, \quad k \geq 0,$$

be the size of the most populated generation in the $(k+1)$ -th cycle. Notice that the sequence $(\sigma_k - \sigma_{k-1})_{k \geq 0}$ as well as the sequence of pairs $(S_k, M_k)_{k \geq 0}$ are i.i.d. under P_0^V . The following asymptotic results hold under Assumption 1.1 (see, for instance, Theorem 2.1, Theorem 2.2, and Lemma 8.1 in [9], respectively):

$$\lim_{n \rightarrow \infty} n^{\delta/2} P_0^V(S_0 > n) = K_0 \in (0, \infty), \quad (12)$$

$$\lim_{n \rightarrow \infty} n^\delta P_0^V(\sigma_0 > n) = K_1 \in (0, \infty), \quad (13)$$

$$\lim_{n \rightarrow \infty} n^\delta P_0^V(M_0 > n) = K_2 \in (0, \infty). \quad (14)$$

3 Proof of the main results

We begin with a brief outline of the proof. First, using properties of the regular variation and the tail asymptotic of the renewal times which is given by (13), we reduce the study of ξ_n^* to that of $\xi_{T_n}^*$. The bounds for the latter sequence, stated in (8), enable us then to exploit the connection between the random walk and the branching process V introduced in Section 2. We remark that a similar strategy has been used for RWRE in [8]. The implementation of this approach (which at most stages is technically fairly different in this paper from the one presented in [8]) is based on the existence of the renewal structure (life cycles) for the branching process and the asymptotic results for the distribution tails of the key random variables stated in (12)-(14) above.

We start the proof with the following 0 – 1 law for the maximal occupation time of the random walk. A similar statement for one-dimensional random walks in random environment is given in [8, Proposition 3.1].

Lemma 3.1. *Let $\phi : \mathbb{N} \rightarrow \mathbb{R}^+$ be an unbounded eventually increasing function. Then*

$$\limsup_{n \rightarrow \infty} \frac{\xi_n^*}{\phi(n)} = K_4 \in [0, \infty] \quad P_0 - \text{a. s.}$$

Proof. Fix any constant $c \geq 0$ and for any realization X of the random walk let

$$g_c(X) := \mathbf{1}_{\{\limsup_{n \rightarrow \infty} \frac{\xi_n^*}{\phi(n)} = c\}},$$

where the random variables ξ_n^* are computed along the infinite path X of the random walk.

Following [12] (see the paragraph right before Lemma 3 there), we next define recursively a sequence of random times $(\eta_{k,m})_{m \in \mathbb{N}, k \in \mathbb{Z}}$ by setting

$$\eta_{k,0} := -1 \quad \text{and} \quad \eta_{k,m+1} := \inf\{n > \eta_{k,m} : X_n \geq k\}.$$

For a fixed $k \in \mathbb{N}$, the sequence $\eta_{k,m}$ represents the successive times when the random walk is located on the right-hand side of the site $k - 1$. Thus the sequence $X^{(k)} = (X_{\eta_{k,m}})_{m \geq 0}$ extracts from the path of the random walk the fragments which are included in the half-line $\{x \in \mathbb{N} : x \geq k\}$. Notice that for a fixed $k \in \mathbb{N}$, random variables $\eta_{k,m}$ are stopping times with respect to the natural filtration of X under the law $P_{0,\omega}$.

Let G_k be the σ -algebra generated by the sequence $(X^{(n)})_{n \geq k}$, that is

$$G_k = \sigma(X^{(k)}, X^{(k+1)}, \dots).$$

It is formally shown in [12, Lemma 3] that the quenched distribution of $X^{(k)}$ in a fixed cookie environment ω is independent of $(\omega_i)_{i \leq k-1}$. Recall now the coin-tossing construction of the excited random walks which is described in (9) and (10) above (see [10, Section 4] or [9, Section 2] for more details). By virtue of (10), in terms of the “coin variables” introduced in (9), we have

$$G_k \subset \sigma(B^{(k)}, B^{(k+1)}, B^{(k+2)}, \dots), \quad k \geq 0, \quad (15)$$

where $B^{(k)} = (B(k, i))_{i \in \mathbb{N}}$.

Under Assumption 1.1, both X_n and the location of the most visited by time n site (say, the right-most one if there are several such equally visited sites) are transient to the right. In particular, since $\max_{x < k} \xi_n(x)$ is bounded above by the total time spent by the random walk on the half-line $\{z \in \mathbb{Z} : z < k\}$, the random variable $g_c(X)$ is measurable with respect to $\sigma(B^{(k)}, B^{(k+1)}, B^{(k+2)}, \dots)$ for any $k \geq 0$. Since $B^{(k)}$ are independent under $P_{0,\omega}$, Kolmogorov’s 0 – 1 law for independent sequences implies that for \mathbb{P} -almost every fixed cookie environment $\omega \in \Omega_M$, the random variable $g_c(X)$ considered as a function of the random walk’s path is an $P_{0,\omega}$ -almost sure constant (compare with the first step in the proof of [8, Proposition 3.1]).

Therefore, we can without loss of generality consider $g_c(X)$ as a function of ω only (but not of the outcome of the “coin tossing” procedure) and correspondingly denote it by, say, $g_c(\omega)$. Specifically, the values of $g_c(\omega)$ can be chosen in such a way that $P_{0,\omega}(g_c(X) = g_c(\omega)) = 1$ for \mathbb{P} -almost every ω . In view of (15), this implies that for any $k \geq 0$,

$$g_c(\omega) \in \sigma(\omega_k, \omega_{k+1}, \dots)$$

In other words, $g_c(\omega)$ is translation-invariant with respect to the shift operator $\theta : \Omega_M \rightarrow \Omega_M$ such that $(\theta\omega)_n = \omega_{n+1}$, $n \in \mathbb{Z}$. By virtue of condition (a) of Assumption 1.1, Kolmogorov's 0–1 law implies then that $g_c(\omega)$ is a \mathbb{P} –a. s. constant function for any $c \geq 0$. Since $g_c(\omega)$ are indicator functions taking values 0 and 1 only, this completes the proof of the lemma. \square

3.1 Non-ballistic regime: Proof of Theorem 1.2

Part (i). We first consider the case when $\delta < 2$. The remaining case $\delta = 2$ is exceptional and is treated separately in Lemma 3.3 below.

Lemma 3.2. *Assumption 1.1 implies (5) when $\delta \in (1, 2)$.*

Proof of Lemma 3.2. In order to prove the claim of the lemma, it suffices to show that for all $\delta \in (1, 2)$ and any constant $c > 0$ which is small enough we have

$$\liminf_{n \rightarrow \infty} P_0(\xi_n^* \geq c\sqrt{n}) > 0. \quad (16)$$

Indeed, (16) together with the reverse Fatou's lemma imply that

$$\begin{aligned} P_0\left(\limsup_{n \rightarrow \infty} \frac{\xi_n^*}{\sqrt{n}} > c\right) &\geq P_0\left(\limsup_{m \rightarrow \infty} \frac{\xi_{T_m}^*}{T_m^{1/2}} > c\right) = E_0\left(\limsup_{m \rightarrow \infty} \mathbf{1}_{\{\xi_{T_m}^* > cT_m^{1/2}\}}\right) \\ &\geq \limsup_{m \rightarrow \infty} E_0(\mathbf{1}_{\{\xi_{T_m}^* > cT_m^{1/2}\}}) = \limsup_{m \rightarrow \infty} P_0(\xi_{T_m}^* > cT_m^{1/2}) > 0. \end{aligned}$$

By virtue of Lemma 3.1, this yields the claim.

We now turn to the proof that (16) holds for any sufficiently small $c > 0$. Observe that according to (7) and (8), we have

$$\begin{aligned} P_0((\xi_{T_m}^*)^2 \geq c^2 T_m) &= P_0\left(\left(\max_{k \leq m} D_k^m\right)^2 \geq c^2 \left(m + 2 \sum_{k \leq m} D_k^m\right)\right) \\ &\geq P_0^V\left(\max_{0 \leq i < \varrho_m} M_i^2 \geq c^2 \left(m + 2 \sum_{i=0}^{\varrho_m} S_i\right)\right). \end{aligned}$$

Denote by $\mu := E_0^V[\sigma_0]$ the annealed (i. e., taken under the law P_0^V) expectation of σ_0 , choose an arbitrary $\delta' \in (1, \delta)$, and define

$$a_m^\pm = \lfloor \mu^{-1}(m \pm m^{1/\delta'}) \rfloor, \quad (17)$$

where we use the notation $\lfloor x \rfloor$ to denote the integer part of a real number x . It follows from the last inequality stated above that

$$\begin{aligned} P_0((\xi_{T_m}^*)^2 \geq c^2 T_m) &\quad (18) \\ &\geq P_0^V\left(\max_{1 \leq i \leq a_m^-} M_i^2 \geq c^2 \left(m + 2 \sum_{i=1}^{a_m^+ + 1} S_i\right)\right) - P_0^V(\varrho_m > a_m^+) - P_0^V(\varrho_m < a_m^-). \end{aligned}$$

Since $\{\varrho_m \geq a_m^+\} = \{m \geq \sigma_{a_m^+}\} = \{m \geq \sum_{i=0}^{a_m^+} (\sigma_i - \sigma_{i-1})\}$, then

$$P_0^V(\varrho_m > a_m^+) \leq P_0^V\left(\frac{\sum_{i=0}^{a_m^+} (\sigma_i - \sigma_{i-1} - \mu)}{(a_m^+)^{1/\delta}} \leq -\frac{m^{1/\delta'}}{(a_m^+)^{1/\delta}}\right). \quad (19)$$

Hence (13) and a stable limit theorem for i.i.d. random variables $\sigma_i - \sigma_{i-1}$ (see, for instance, Theorem 1.5.1 in [5]) imply that $\lim_{m \rightarrow \infty} P_0^V(\varrho_m > a_m^+) = 0$. Similarly, one can show that $\lim_{m \rightarrow \infty} P_0^V(\varrho_m < a_m^-) = 0$. Therefore, in order to prove (16), it suffices to show that the following strict lower bound holds:

$$\liminf_{m \rightarrow \infty} P_0^V\left(\max_{1 \leq i \leq a_m^-} M_i^2 \geq c^2\left(m + 2 \sum_{i=1}^{a_m^++1} S_i\right)\right) > 0. \quad (20)$$

Toward this end, fix any positive constants $\beta, \gamma > 0$ such that

$$\frac{1}{2} \cdot (K_2 \cdot \beta^{-\delta/2} - K_0 \cdot \gamma^{-\delta/2}) - K_2^2 \cdot \beta^{-\delta} > 0, \quad (21)$$

and observe that

$$\begin{aligned} P_0^V\left(\max_{1 \leq i \leq a_m^-} M_i^2 \geq c^2\left(m + 2 \sum_{i=1}^{a_m^++1} S_i\right)\right) &\geq \\ &\geq P_0^V\left(\max_{1 \leq i \leq a_m^-} M_i^2 \geq \beta m^{2/\delta}, m + 2 \sum_{i=1}^{a_m^++1} S_i \leq \frac{\beta m^{2/\delta}}{c^2}\right) \geq P_0^V\left(\bigcup_{i=1}^{a_m^-} A_{i,m}\right), \end{aligned}$$

where

$$A_{i,m} := \left\{M_i^2 \geq \beta m^{2/\delta}, S_i \leq \gamma m^{2/\delta}, m + 2 \sum_{\substack{1 \leq j \leq a_m^++1, \\ j \neq i}} S_j < (\beta/c^2 - \gamma)m^{2/\delta}\right\}. \quad (22)$$

Therefore, the inclusion-exclusion formula yields

$$\begin{aligned} P_0^V\left(\max_{1 \leq i \leq a_m^-} M_i^2 \geq c^2\left(m + 2 \sum_{i=1}^{a_m^++1} S_i\right)\right) &\geq P_0^V\left(\bigcup_{i=1}^{a_m^-} A_{i,m}\right) \\ &\geq \sum_{i=1}^{a_m^-} P_0^V(A_{i,m}) - \sum_{i=1}^{a_m^-} \sum_{j=i+1}^{a_m^-} P_0^V(A_{i,m} \cap A_{j,m}) \\ &\geq a_m^- P_0^V(A_{1,m}) - (a_m^-)^2 P_0^V(M_1^2 \geq \beta m^{2/\delta}, M_2^2 \geq \beta m^{2/\delta}). \end{aligned} \quad (23)$$

Using the independence of the life-cycles of the underlying branching process, we obtain that

$$P_0^V(A_{1,m}) = P_0^V(M_1^2 \geq \beta m^{2/\delta}, S_1 \leq \gamma m^{2/\delta}) \cdot P_0^V\left(m + \sum_{j=2}^{a_m^++1} S_j < (\beta/c^2 - \gamma)m^{2/\delta}\right).$$

Taking into account (12) and (14), one can deduce from the following inequality:

$$P_0^V(M_1^2 \geq \beta m^{2/\delta}, S_1 \leq \gamma m^{2/\delta}) \geq P_0^V(M_1^2 \geq \beta m^{2/\delta}) - P_0^V(S_1 > \gamma m^{2/\delta}),$$

that

$$\liminf_{m \rightarrow \infty} a_m^- \cdot P_0^V(M_1^2 \geq \beta m^{2/\delta}, S_1 \leq \gamma m^{2/\delta}) \geq K_2 \cdot \beta^{-\delta/2} - K_0 \cdot \gamma^{-\delta/2}.$$

Furthermore, it follows from (12) and a stable limit theorem for i.i.d. variables S_i (see, for instance, Theorem 1.5.1 in [5]) that the following limit exists and is strictly positive:

$$\lambda(c, \beta, \gamma) := \lim_{m \rightarrow \infty} P_0^V\left(m + 2 \sum_{j=2}^{a_m^+ + 1} S_j < (\beta/c^2 - \gamma)m^{2/\delta}\right) > 0.$$

Moreover, given $\beta, \gamma > 0$, we can choose $c > 0$ so small that $\lambda(c, \beta, \gamma) > 1/2$. For such a constant $c > 0$, (23) along with (14) yield

$$\begin{aligned} \liminf_{m \rightarrow \infty} P_0^V\left(\max_{1 \leq i \leq a_m^-} M_i^2 \geq c^2 \left(m + 2 \sum_{i=1}^{a_m^+ + 1} S_i\right)\right) \\ \geq \lambda(c, \beta, \gamma) \cdot (K_2 \cdot \beta^{-\delta/2} - K_0 \cdot \gamma^{-\delta/2}) - K_2^2 \cdot \beta^{-\delta} \\ \geq \frac{1}{2} \cdot (K_2 \cdot \beta^{-\delta/2} - K_0 \cdot \gamma^{-\delta/2}) - K_2^2 \cdot \beta^{-\delta}. \end{aligned}$$

In view of (21), this implies (20), and hence (16) for the indicated above values of the parameter $c > 0$. \square

Lemma 3.2 yields part (i) of Theorem 1.2 for $\delta \in (0, 2)$. The remaining case $\delta = 2$ is considered in the next lemma, by using a slightly different approach which is based on a modification of an argument in [8], cf. [8, p. 172].

Lemma 3.3. *Let Assumption 1.1 hold and suppose that $\delta = 2$. Then (5) holds along random subsequence $(T_m)_{m \geq 1}$ of integers n . That is,*

$$\limsup_{m \rightarrow \infty} \frac{\xi_{T_m}^*}{T_m^{1/2}} > 0, \quad P_0 - \text{a. s.}$$

Proof of Lemma 3.3. Recall (11) and (17). The proof of the lemma relies on the Borel-Cantelli argument which is applied to the following version of (18):

$$\begin{aligned} P_0((\xi_{T_m}^*)^2 \geq c^2 T_m \text{ i. o.}) &\geq P_0^V\left(\max_{1 \leq i \leq a_m^-} M_i^2 \geq c^2 \left(m + 2 \sum_{i=1}^{a_m^+ + 1} S_i\right) \text{ i. o.}\right) \\ &\quad - P_0^V(\varrho_m > a_m^+ \text{ i. o.}) - P_0^V(\varrho_m < a_m^- \text{ i. o.}) \end{aligned} \quad (24)$$

with a suitably chosen subsequence of the integer indices m .

First, we will establish certain large deviation type estimates for the distribution tails of the random variables ϱ_m . To this end, observe that the inequality stated in (19) remains

true for any $\delta > 1$, in particular for $\delta = 2$. Furthermore, (13) and the large deviation estimate stated, for instance, in [5, Theorem 3.4.1] imply that the following holds under Assumption 1.1 (with arbitrary $\delta > 1$) for a suitable constant $c_1 = c_1(\delta) > 0$:

$$P_0^V(\varrho_m > a_m^+) \leq \frac{c_1 a_m^+}{m^{\delta/\delta'}}. \quad (25)$$

Similarly, using the following inequality instead of (19):

$$P_0^V(\varrho_m \leq a_m^-) = P_0^V(m \leq \sigma_{a_m^-}) \leq P_0^V\left(\frac{\sum_{i=0}^{a_m^-} (\sigma_i - \sigma_{i-1} - \mu)}{(a_m^-)^{1/\delta}} \geq \frac{m^{1/\delta'}}{(a_m^-)^{1/\delta}}\right), \quad (26)$$

one can deduce from (13) and [5, Theorem 3.4.1] that the following holds under Assumption 1.1 (with arbitrary $\delta > 1$) for some constant $c_2 = c_2(\delta) > 0$:

$$P_0^V(\varrho_m < a_m^-) \leq \frac{c_2 a_m^-}{m^{\delta/\delta'}}. \quad (27)$$

We remark that, in the course of proving (25), in order to formally meet the lower tail conditions of Theorem 3.4.1 in [5] one can, for instance, use in (19) the following “unpolarized” version of $\sigma_i - \sigma_{i-1} - \mu$ which has the same structure of upper and lower distribution tails:

$$(\sigma_i - \sigma_{i-1} - \mu)' := U_i \cdot (\sigma_i - \sigma_{i-1} - \mu),$$

where $U = (U_i)_{i \geq 0}$ is a sequence of i.i.d. Bernoulli random variables, independent of “anything else” (i. e., such that the probability law P^U of U is independent of the measure P^V in the enlarged probability space), and such that

$$P^U(U_i = 1) = P^U(U_i = -1) = \frac{1}{2}.$$

Notice that (cf. (19))

$$P_0^V\left(\sum_{i=0}^{a_m^+} (\sigma_i - \sigma_{i-1} - \mu) \leq -m^{1/\delta'}\right) \leq P_0^V\left(\sum_{i=0}^{a_m^+} (\sigma_i - \sigma_{i-1} - \mu)' \leq -m^{1/\delta'}\right).$$

Fix now any $c > 0$ and recall (18). For $k \in \mathbb{N}$, let $m_k = k^k$, $n_k = 2m_{k-1}$, and define the following events:

$$\begin{aligned} B_k &= \left\{ \max_{a_{n_k} < i \leq a_{m_k}^-} M_i^2 \geq c^2 \left(m + 2 \sum_{i=1}^{a_{m_k}^+ + 1} S_i \right) \right\}, \\ D_k &= \left\{ c^2 \left(m + 2 \sum_{i=1}^{a_{m_{k-1}}^+ + 1} S_i \right) > m_k \log m_k \right\}, \\ E_k &= \left\{ \max_{a_{n_k} < i \leq a_{m_k}^-} M_i^2 \geq 2m_k \log m_k, c^2 \left(m + 2 \sum_{i=a_{m_{k-1}}^+ + 2}^{a_{m_k}^+ + 1} S_i \right) \leq m_k \log m_k \right\}. \end{aligned}$$

By virtue of (12), Theorem 3.8.1 along with Theorem 1.5.1-(ii) in [5] (notice that the random centering required in the former is given by the latter), there exists a positive constant $c_3 > 0$ such that

$$P_0^V(D_k) \leq c_3 \frac{m_{k-1}}{m_k \log m_k} \leq c_3 \frac{(k-1)^{k-1}}{k^{k+1} \log k} \leq c_3 k^{-2}.$$

Thus by the first Borel-Cantelli lemma, $P_0^V(D_k \text{ i.o.}) = 0$. On the other hand, similarly to (22) and (23), setting

$$\tilde{A}_{i,k} := \left\{ M_i^2 \geq 2m_k \log m_k, \ c^2 \left(m + 2 \sum_{j=2}^{a_{m_k}^+ - a_{m_k}^-} S_j \right) < m_k \log m_k \right\},$$

we obtain

$$\begin{aligned} P_0^V(E_k) &\geq P_0^V\left(\bigcup_{i=1}^{a_{m_k}^-} \tilde{A}_{i,k}\right) \geq \sum_{i=1}^{a_{m_k}^- - a_{n_k}} P_0^V(\tilde{A}_{1,k}) - \sum_{i=1}^{a_{m_k}^-} \sum_{j=i+1}^{a_{m_k}^-} P_0^V(\tilde{A}_{i,k} \cap \tilde{A}_{j,k}) \\ &\geq (a_{m_k}^- - a_{n_k}) P_0^V(\tilde{A}_{1,k}) - (a_{m_k}^-)^2 P_0^V(M_1^2 \geq m_k \log m_k, M_2^2 \geq m_k \log m_k). \end{aligned}$$

Thus by virtue of (14) and Theorem 1.5.1-(ii) in [5], if $c > 0$ is sufficiently small, then

$$P(E_k) \geq \frac{c_4}{\log m_k} = \frac{c_4}{k \log k},$$

for a suitable constant $c_4 > 0$. Since the events E_k are independent of each other, the second Borel-Cantelli lemma yields $P_0^V(E_k \text{ i.o.}) = 1$.

For an event A in the underlying probability space, let A^c denote its complement. Observe now that

$$E_k \cap D_k^c \subset B_k, \quad k \geq 0,$$

and hence $P_0^V(B_k \text{ i.o.}) = 1$ for any constant $c > 0$ small enough. Finally, (25) and (27) imply that

$$P_0^V(\varrho_{m_k} < a_{m_k}^- \text{ i.o.}) = P_0^V(\varrho_{m_k} > a_{m_k}^+ \text{ i.o.}) = 0.$$

It thus follows from (24) that

$$P_0(\xi_n^* \geq c\sqrt{n} \text{ i.o.}) \geq P_0((\xi_{T_m}^*)^2 \geq c^2 T_m \text{ i.o.}) = 1$$

for sufficiently small values of the constant $c > 0$. This completes the proof of the lemma. \square

In view of Lemmas 3.2 and 3.3, the proof of part (i) of Theorem 1.2 is completed. \square

Part (ii). First, we will show that for any constant $\alpha > 1/\delta$,

$$\lim_{n \rightarrow \infty} \frac{\xi_n^*}{n^{1/2}(\log n)^\alpha} = 0, \quad P_0 - \text{a.s.}$$

Fix any $\alpha > 1/\delta$ and let $\phi(n) = n^{1/2}(\log n)^\alpha$. Recall the alternative notation $\xi^*(n)$ for ξ_n^* . Observe that in order to prove the above claim, it suffices to show that

$$\limsup_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} = 0, \quad P_0 - \text{a.s.} \quad (28)$$

Indeed, let $k_m, m \in \mathbb{N}$, be the (uniquely defined) non-negative random integers such that

$$T_{k_m} < m \leq T_{k_m+1}, \quad m \in \mathbb{N}. \quad (29)$$

Then, since $\phi(n)$ is an eventually increasing sequence,

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{\xi^*(m)}{\phi(m)} &\leq \limsup_{m \rightarrow \infty} \frac{\xi^*(T_{k_m+1})}{\phi(m)} \leq \limsup_{m \rightarrow \infty} \frac{\xi^*(T_{k_m+1})}{\phi(T_{k_m})} \\ &\leq \limsup_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} = 0, \quad P_0 - \text{a.s.} \end{aligned} \quad (30)$$

Recall now (25) and (27). Let $\psi_\varepsilon(m) = m^{2/\delta}(\log m)^{-\varepsilon}$ with $\varepsilon \in (0, 2\alpha)$ and let $m_i = 3^i$ for $i \in \mathbb{N}$. Then, for any constant $c > 0$, we have

$$\begin{aligned} \sum_{i=1}^{\infty} P_0(T_{m_i} < c\psi_\varepsilon(m_{i+1})) &\leq \sum_{i=1}^{\infty} \left[P_0^V \left(2 \sum_{k=0}^{a_{m_i}^- - 1} S_k < c\psi_\varepsilon(m_{i+1}) \right) + P_0^V(\varrho_{m_i} < a_{m_i}^-) \right] \\ &\leq \sum_{i=1}^{\infty} \left[P_0^V \left(2 \max_{0 \leq k < a_{m_i}^-} S_k < c\psi_\varepsilon(m_{i+1}) \right) + P_0^V(\varrho_{m_i} < a_{m_i}^-) \right] \\ &\leq \sum_{i=1}^{\infty} \left[\left(1 - \frac{c_5}{(c\psi_\varepsilon(m_{i+1}))^{\delta/2}} \right)^{a_{m_i}^-} + \frac{c_2 a_{m_i}^-}{(m_i)^{\delta/\delta'}} \right] \leq \sum_{i=1}^{\infty} c_6 \cdot \left[e^{-c_7(i+1)\varepsilon^\delta} + e^{-c_8 i} \right] < \infty, \end{aligned} \quad (31)$$

where $c_5, c_5, c_7, c_8 > 0$ are suitable positive constants and c_2 is the constant which appears at (27). Thus, by the Borel-Contelli lemma,

$$\liminf_{i \rightarrow \infty} \frac{T_{m_i}}{\psi_\varepsilon(m_{i+1})} = \infty, \quad P_0 - \text{a.s.}$$

Since for each $n \in \mathbb{N}$,

$$m_i < n \leq m_{i+1} \quad \text{for some } i \in \mathbb{N} \text{ which is uniquely determined by } n, \quad (32)$$

then

$$\liminf_{n \rightarrow \infty} \frac{T_n}{\psi_\varepsilon(n)} \geq \liminf_{i \rightarrow \infty} \frac{T_{m_i}}{\psi_\varepsilon(m_{i+1})} = \infty, \quad P_0 - \text{a.s.}, \quad (33)$$

and hence $P_0(T_m < \psi_\varepsilon(m) \text{ i. o.}) = 0$. This implies that for any constant $b > 0$,

$$\begin{aligned} P_0\left(\limsup_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} > b\right) &\leq P_0\left(\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq k \leq m} D_k^m}{\phi(\psi_\varepsilon(m-1))} > \frac{b}{2}\right) \\ &\leq P_0^V\left(\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq k \leq \varrho_m} M_k}{\phi(\psi_\varepsilon(m-1))} > \frac{b}{2}\right) \leq P_0^V\left(\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq k \leq m} M_k}{\phi(\psi_\varepsilon(m-1))} > \frac{b}{2}\right) \\ &= P_0^V\left(\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq k \leq m} M_k}{m^{1/\delta}(\log m)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}}\right), \end{aligned} \quad (34)$$

where in the last but one step we used the inequality $\varrho_m \leq m$. Let $r_m \in [0, m]$ be the largest integer such that $\max_{0 \leq k \leq m} M_k = M_{r_m}$. That is

$$r_m = \max\{n \in [0, m] : M_n = \max_{0 \leq k \leq m} M_k\}, \quad m \in \mathbb{N}. \quad (35)$$

Note that $r_m \leq m$ for any $m \in \mathbb{N}$. Therefore,

$$\begin{aligned} P_0^V\left(\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq k \leq m} M_k}{m^{1/\delta}(\log m)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}}\right) &= P_0^V\left(\limsup_{m \rightarrow \infty} \frac{M_{r_m}}{m^{1/\delta}(\log m)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}}\right) \\ &\leq P_0^V\left(\limsup_{m \rightarrow \infty} \frac{M_{r_m}}{r_m^{1/\delta}(\log r_m)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}}\right) \\ &\leq P_0^V\left(\limsup_{m \rightarrow \infty} \frac{M_m}{m^{1/\delta}(\log m)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}}\right). \end{aligned} \quad (36)$$

It follows from (14) that if $\varepsilon \in (0, 2\alpha)$ is chosen in such a way that in fact $\alpha - \varepsilon/2 > 1/\delta$, then

$$\sum_{m=1}^{\infty} P_0^V\left(\frac{M_{m_i}}{m_i^{1/\delta}(\log m_i)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}}\right) < \infty.$$

Thus the Borel-Cantelli lemma combined with (32) implies that for $\varepsilon \in (0, 2\alpha - 2/\delta)$,

$$P_0^V\left(\frac{M_m}{m^{1/\delta}(\log m)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}} \text{ i. o.}\right) = P_0^V\left(\frac{M_{m_i}}{m_i^{1/\delta}(\log m_i)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}} \text{ i. o.}\right) = 0.$$

It then follows from (34) and (36) that

$$P_0\left(\limsup_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} > b\right) \leq P_0^V\left(\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq k \leq m} M_k}{\phi(\psi_\varepsilon(m-1))} > \frac{b}{2}\right) = 0.$$

This completes the proof of (28).

We now turn to the proof that for any constant $\alpha > 1/\delta$,

$$\lim_{n \rightarrow \infty} \frac{(\log n)^\alpha \xi_n^*}{n^{1/2}} = \infty, \quad P_0 - \text{a. s.}$$

Fix any $\alpha > 1/\delta$ and let $\phi(n) = n^{1/2}(\log n)^{-\alpha}$. In order to prove the claim, it suffices to show that

$$\liminf_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(T_{m+1})} = \infty, \quad P_0 - \text{a. s.} \quad (37)$$

Indeed, in view of (29), it follows from (37) that

$$\infty = \liminf_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(T_{m+1})} \leq \liminf_{m \rightarrow \infty} \frac{\xi^*(T_{k_m})}{\phi(T_{k_m+1})} \leq \liminf_{m \rightarrow \infty} \frac{\xi^*(m)}{\phi(m)}, \quad P_0 - \text{a. s.}$$

We will next show that (37) indeed holds true. Fix any $\varepsilon \in (0, \alpha\delta - 1)$ and define

$$\psi_\varepsilon(m) := m^{2/\delta}(\log m)^{2/\delta+\varepsilon}, \quad m \in \mathbb{N}.$$

By a counterpart of the law of the iterated logarithm for i.i.d. random variables in the domain of attraction of a stable law and with infinite variance, we have (see, for instance, Theorem 1.6.6. in [5]):

$$\limsup_{m \rightarrow \infty} \frac{\sum_{k=1}^m S_k}{\psi_\varepsilon(m)} = 0, \quad P_0^V - \text{a. s.}$$

Combining this result with the fact that $\lim_{m \rightarrow \infty} P_0^V(\frac{\varrho_m}{m} = \mu^{-1}) = 1$ (which is an implication of the renewal theorem applied to the renewal sequence σ_k), we obtain that

$$P_0(T_m > \psi_\varepsilon(m) \text{ i. o.}) = 0.$$

Therefore,

$$\liminf_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(\psi_\varepsilon(m+1))} \leq \liminf_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(T_{m+1})}, \quad P_0 - \text{a. s.}$$

Thus it suffices to prove that the left-hand side of the above inequality is infinity. Recall (17) and (27). Then, for $m_i = 3^i$ and any constant $c > 0$, similarly to (31), we have:

$$\begin{aligned} \sum_{i=1}^{\infty} P_0\left(\xi^*(T_{m_i}) < c\phi(\psi_\varepsilon(m_{i+1}+1))\right) &\leq \sum_{i=1}^{\infty} P_0^V\left(\max_{0 \leq k < \varrho_{m_i}} M_k < c\phi(\psi_\varepsilon(m_{i+1}+1))\right) \\ &\leq \sum_{i=1}^{\infty} \left[P_0^V\left(\max_{0 \leq k < a_{m_i}^-} M_k < c\phi(\psi_\varepsilon(m_{i+1}+1))\right) + P_0^V(a_{m_i}^- > \varrho_{m_i}) \right] \\ &\leq \sum_{i=1}^{\infty} c_9 \cdot \left[e^{-c_{10}(i+1)^{\alpha\delta-1-\varepsilon}} + \frac{a_{m_i}^-}{(m_i)^{\delta/\delta'}} \right] < \infty, \end{aligned} \quad (38)$$

where $c_9 > 0$ and $c_{10} > 0$ are some appropriate positive constants. Therefore, the Borel-Contelli lemma yields (recall that the value of the parameter ε is chosen from the interval $(0, \alpha\delta - 1)$, and hence $\alpha\delta - 1 - \varepsilon > 0$):

$$\liminf_{i \rightarrow \infty} \frac{\xi^*(T_{m_i})}{\phi(\psi_\varepsilon(m_{i+1}+1))} = \infty, \quad P_0 - \text{a. s.}$$

This completes the proof of (37) by using a suitable variation of (33). \square

3.2 Ballistic regime: Proof of Theorem 1.3

By [4, Theorem 1.5.3] combined with [4, Theorem 1.9.5], the regularly varying sequence $\phi(n)$ is asymptotically equivalent to a non-decreasing sequence $\bar{\phi}(n)$ which is also regularly varying with index ρ . By the asymptotic equivalence we mean that $\lim_{n \rightarrow \infty} \bar{\phi}(n)/\phi(n) = 1$. Since

$$\limsup_{n \rightarrow \infty} \frac{\xi_n^*}{\phi(n)} = \limsup_{n \rightarrow \infty} \frac{\xi_n^*}{\bar{\phi}(n)} \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\xi_n^*}{\phi(n)} = \liminf_{n \rightarrow \infty} \frac{\xi_n^*}{\bar{\phi}(n)},$$

we can assume without loss of generality that $\phi(n)$ is a non-decreasing sequence. In the rest of the paper we will make this additional assumption without further notice.

Part (i). Suppose first that

$$\sum_{m=1}^{\infty} \frac{1}{[\phi(m)]^\delta} < \infty. \quad (39)$$

Since $\varrho_m \leq m$ and $m \leq T_m$, then for any constant $b > 0$ we have

$$\begin{aligned} P_0\left(\limsup_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} > b\right) &\leq P_0\left(\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq k \leq m} D_k^m}{\phi(m-1)} > \frac{b}{2}\right) \\ &\leq P_0^V\left(\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq k \leq \varrho_m} M_k}{\phi(m-1)} > \frac{b}{2}\right) \leq P_0^V\left(\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq k \leq m} M_k}{\phi(m-1)} > \frac{b}{2}\right). \end{aligned} \quad (40)$$

Recall r_m from (35). Then

$$\begin{aligned} P_0^V\left(\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq k \leq m} M_k}{\phi(m-1)} > \frac{b}{2}\right) &= P_0^V\left(\limsup_{m \rightarrow \infty} \frac{M_{r_m}}{\phi(m)} > \frac{b}{2}\right) \\ &\leq P_0^V\left(\limsup_{m \rightarrow \infty} \frac{M_{r_m}}{\phi(r_m)} > \frac{b}{2}\right) \leq P_0^V\left(\limsup_{n \rightarrow \infty} \frac{M_n}{\phi(n)} > \frac{b}{2}\right). \end{aligned}$$

It follows from (14) and (39) that

$$\sum_{m=1}^{\infty} P_0^V\left(\frac{M_n}{\phi(n)} > \frac{b}{2}\right) < \infty.$$

Thus the Borel-Cantelli lemma and (40) imply

$$P_0\left(\limsup_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} > b\right) \leq P_0^V\left(\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq k \leq m} M_k}{\phi(m-1)} > \frac{b}{2}\right) = 0.$$

By virtue of (30), this completes the proof of the first half of part (i) of Theorem 1.3.

Suppose now that

$$\sum_{m=1}^{\infty} \frac{1}{[\phi(m)]^\delta} = \infty. \quad (41)$$

Recall (2) and the fact that $v > 0$ (the asymptotic speed is positive) when $\delta > 2$. Observe that in order to prove that

$$\limsup_{n \rightarrow \infty} \frac{\xi_n^*}{\phi(n)} = \infty, \quad P_0 - \text{a. s.},$$

it suffices to show that for any constant $b > 0$,

$$\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq i < \varrho_m} M_k}{\phi(\lfloor 2v^{-1}m \rfloor)} > b, \quad P_0^V - \text{a. s.} \quad (42)$$

Indeed, $P_0(T_m > 2v^{-1}m \text{ i. o.}) = 0$ by virtue of (6). Hence (42) implies that for every $b > 0$,

$$\begin{aligned} P_0\left(\limsup_{m \rightarrow \infty} \frac{\xi^*(m)}{\phi(m)} > b\right) &\geq P_0\left(\limsup_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(T_m)} > b\right) \\ &\geq P_0\left(\limsup_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(\lfloor 2v^{-1}m \rfloor)} > b\right) \geq P_0^V\left(\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq i < \varrho_m} M_i}{\phi(\lfloor 2v^{-1}m \rfloor)} > b\right) = 1. \end{aligned}$$

Thus the proof of part (i) of Theorem 1.3 will be completed once we prove (42).

We turn now to the proof of (42). Recall the notation $\mu = E_0^V[\sigma_0]$ that we have used before. By the renewal theorem $P_0^V(\lim_{m \rightarrow \infty} \frac{\varrho_m}{m} = \frac{1}{\mu}) = 1$, and hence $P_0^V(\varrho_m < \frac{m}{2\mu} \text{ i. o.}) = 0$. It thus suffices to show that

$$\limsup_{m \rightarrow \infty} \frac{\max_{0 \leq i \leq \lfloor m\mu^{-1}/2 \rfloor} M_k}{\phi(\lfloor 2v^{-1}m \rfloor)} \geq \limsup_{m \rightarrow \infty} \frac{M_{\lfloor m\mu^{-1}/2 \rfloor}}{\phi(\lfloor 2v^{-1}m \rfloor)} > b, \quad P_0^V - \text{a. s.} \quad (43)$$

To this end, observe that by virtue of (14) and (41),

$$\sum_{m=1}^{\infty} P_0\left(M_{\lfloor m\mu^{-1}/2 \rfloor} > b\phi(\lfloor 2v^{-1}m \rfloor)\right) = \infty.$$

Therefore (43), and hence also (42), follow from the Borel-Contelli lemma. The proof of part (i) of Theorem 1.3 is completed. \square

Part (ii). Let $\phi(n) = n^{1/\delta}(\log n)^{-\alpha}$ for a fixed constant $\alpha > 1/\delta$. In order to prove the claim, it suffices to verify (37) (see the next two lines below (37)). Toward this end, observe that according to the law of large numbers for T_n stated in (6) we have $P_0(T_m > 2mv^{-1} \text{ i. o.}) = 0$, and hence

$$\liminf_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(2v^{-1}(m+1))} \leq \liminf_{m \rightarrow \infty} \frac{\xi^*(T_m)}{\phi(T_{m+1})}, \quad P_0 - \text{a. s.}$$

Thus it suffices to show that the left-hand side of the above inequality is infinity. Let a_m^- be as defined in (17) with the only exception that this time we will use an arbitrary constant $\delta' \in (1, 2)$. In view of (13) and (26), Chebyshev's inequality implies

$$P_0^V(a_m^- > \varrho_m) \leq \frac{a_m^- + 1}{m^{2/\delta'}} \cdot E_0^V[(\sigma_0 - \mu)^2]. \quad (44)$$

Therefore, a slight modification of (38) (namely, formally replacing there the composition of two functions $\phi \circ \psi_\varepsilon$ by the “new” $\phi(n) = n^{1/\delta}(\log n)^{-\alpha}$ and also using (44) instead of (27)) along with the Borel-Contelli lemma imply that $\liminf_{i \rightarrow \infty} \frac{\xi^*(T_{m_i})}{\phi(m_{i+1}+1)} = \infty$, $P_0 - \text{a. s.}$ This completes the proof of (37) by using an appropriate variation of (33). \square

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References

- [1] A.-L. Basdevant and A. Singh, *On the speed of a cookie random walk*, Probab. Theory Related Fields **141** (2008), 625–645.
- [2] A.-L. Basdevant and A. Singh, *Rate of growth of a transient cookie random walk*, Electron. J. Probab. **13** (2008), 811–851.
- [3] I. Benjamini and D. B. Wilson, *Excited random walk*, Electron. Comm. Probab. **8** (2003), 86–92.
- [4] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*. Encyclopedia of Mathematics and its Applications **27**, Cambridge University Press, 1989 (paperback edition).
- [5] A. A. Borovkov and K. A. Borovkov, *Asymptotic Analysis of Random Walks: Heavy-tailed Distributions*, Encyclopedia of Mathematics and its Applications **118**, Cambridge University Press, 2008.
- [6] A. Dembo, N. Gantert, Y. Peres, and Z. Shi, *Valleys and the maximum local time for random walk in random environment*, Probab. Theory and Related Fields **137** (2007), 443–473.
- [7] D. Dolgopyat, *Random walks in one dimensional environment*, lecture notes. Available electronically at <http://www-users.math.umd.edu/~dmitry/RW1d.pdf>.
- [8] N. Gantert and Z. Shi, *Many visits to a single site by a transient random walk in random environment*, Stoch. Proc. Appl. **99** (2002), 159–176.
- [9] E. Kosygina and T. Mountford, *Limit laws of transient excited random walks on integers*, Ann. Inst. H. Poincaré Probab. Statist. **47** (2011), 575–600.
- [10] E. Kosygina and M. Zerner, *Positively and negatively excited random walks on integers, with branching processes*, Electron. J. Prob. **13** (2008), 1952–1979.
- [11] P. Revesz, *Random Walk in Random and Non-Random Environments*, World Scientific Publishing Company, 2005.
- [12] M. Zerner, *Multi-excited random walks on integers*, Probab. Theory Related Fields **133** (2005), 98–122.